Remainder, Factor Theorem and Partial Fractions

Remainder and Factor Theorem

Remainder Theorem
(a) If a polynomial $P(x)$ is divided by a linear divisor $x - c$, the remainder is $P(c)$.
(b) If a polynomial $P(x)$ is divided by a linear divisor $ax + b$, the remainder is $P\left(-\frac{b}{a}\right)$.

Example:
Remainder when $2x^2 + 4x - 1$ is divided by $2x + 1$ is $2\left(-\frac{1}{2}\right)^2 + 4\left(-\frac{1}{2}\right) - 1 = -2\frac{1}{2}$

Factor Theorem
$ax + b$ is a linear factor of the polynomial $P(x)$ if and only if $P\left(-\frac{b}{a}\right) = 0$, i.e. the remainder is 0.

Solving cubic equations
General Steps:
Step 1: Find the first factor by trial-and-error or using the “solve” function of the calculator. Remember to show the working using factor theorem to prove that the first factor is indeed a factor.
Step 2: Use either long division or comparing coefficients to factorise the cubic equation.
Step 3: Equate the original equation to be 0 and solve the equation accordingly.

Special Algebraic Identities
Sum of Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Partial Fractions

General Steps:
Step 1: Check if the fraction is improper or proper.
Step 2: If improper (highest power (degree) of the numerator is equal to or higher than the highest power of the denominator), use long division.

Example:
$\frac{2x^3 + 2}{x^3 - 5x^2 + 2x - 13} \rightarrow$ Improper (degrees are the same)
$\frac{2x^3 + 2}{2x^3 + 2} \rightarrow$ Improper (denominator has higher degree)
$\frac{2x^3 + 2}{4x^3 - 6x + 16} \rightarrow$ Proper (numerator has lower degree)

Step 3: Factorise the base as much as possible

Example:
$\frac{4x}{x^2 - 2} = \frac{4x}{(x + 2)(x - 2)}$

Step 3: Express in partial fractions, according to the table below.

<table>
<thead>
<tr>
<th>Case</th>
<th>Denominator Contains</th>
<th>Algebraic Fraction</th>
<th>Partial Fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Distinct linear factors</td>
<td>$\frac{px + q}{(ax + b)(cx + d)}$</td>
<td>$\frac{A}{ax + b} + \frac{B}{cx + d}$</td>
</tr>
<tr>
<td>2</td>
<td>Repeated linear factors</td>
<td>$\frac{px + q}{(ax + b)^2}$</td>
<td>$\frac{A}{ax + b} + \frac{B}{(ax + b)^2}$</td>
</tr>
<tr>
<td>3</td>
<td>Quadratic factor that cannot be factorised</td>
<td>$\frac{px + q}{(ax + b)(x^2 + c^2)}$</td>
<td>$\frac{A}{ax + b} + \frac{Bx + C}{x^2 + c^2}$</td>
</tr>
</tbody>
</table>

Step 4: Solve the partial fraction by either cover up rule or comparing coefficients or any other method.